

1.3

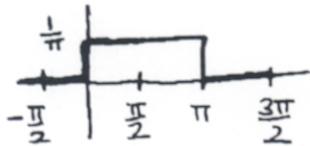
a.)  $p(\theta)$  is a constant (from test)

$$\int_0^{\pi} p(\theta) d\theta = 1 \quad (\text{the needle should be somewhere})$$

$$\Rightarrow \int_0^{\pi} p(\theta) d\theta = \int_0^{\pi} c d\theta = c\theta \Big|_0^{\pi} = c\pi = 1$$

$$\Rightarrow c = \frac{1}{\pi}$$

$$\begin{cases} p(\theta) = \frac{1}{\pi} & 0 < \theta < \pi \\ p(\theta) = 0 & \text{otherwise} \end{cases}$$



$$b) \langle \theta \rangle = \int_0^{\pi} p(\theta) \theta d\theta = \int_0^{\pi} \frac{\theta}{\pi} d\theta = \frac{\theta^2}{2\pi} \Big|_0^{\pi} = \boxed{\frac{\pi^2}{2}}$$

$$\langle \theta^2 \rangle = \int_0^{\pi} p(\theta) \theta^2 d\theta = \int_0^{\pi} \frac{\theta^2}{\pi} d\theta = \frac{\theta^3}{3\pi} \Big|_0^{\pi} = \boxed{\frac{\pi^2}{3}}$$

$$\sigma^2 = \langle \theta^2 \rangle - \langle \theta \rangle^2 = \frac{\pi^2}{3} - \left(\frac{\pi^2}{2}\right)^2 = \pi^2 \left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{\pi^2}{12}}$$

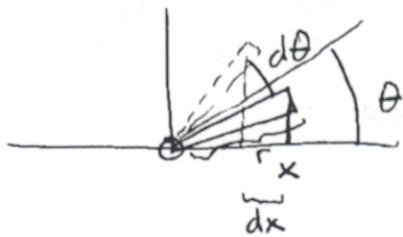
$$c.) \langle \sin \theta \rangle = \int_0^{\pi} \frac{\sin \theta}{\pi} d\theta = -\frac{\cos \theta}{\pi} \Big|_0^{\pi} = \frac{1}{\pi} (1 + 1) = \boxed{\frac{2}{\pi}}$$

$$\langle \cos \theta \rangle = \int_0^{\pi} \frac{\cos \theta}{\pi} d\theta = \frac{\sin \theta}{\pi} \Big|_0^{\pi} = \boxed{0}$$

$$\langle \cos^2 \theta \rangle = \int_0^{\pi} \frac{\cos^2 \theta}{\pi} d\theta = \frac{1}{2\pi} \int_0^{\pi} (\cos 2\theta + 1) d\theta = \left( \frac{1}{4\pi} \sin 2\theta + \frac{\theta}{2\pi} \right) \Big|_0^{\pi} = \boxed{\frac{1}{2}}$$

1.4

a.)



2.

$$x = r \cos \theta$$

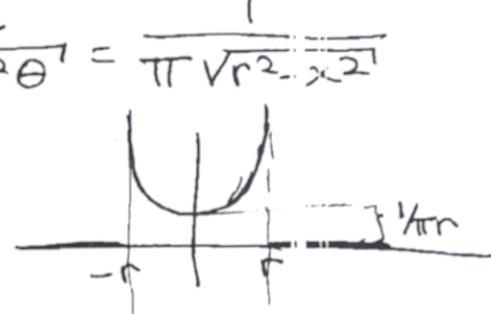
$$dx = r d(\cos \theta) = -r \sin \theta d\theta$$

$$f(\theta) |d\theta| = f(x) |dx|$$

$$\frac{1}{\pi} d\theta = f(x) r \sin \theta d\theta$$

$$f(x) = \frac{1}{\pi r \sin \theta} = \frac{1}{\pi r \sqrt{1 - \cos^2 \theta}} = \frac{1}{\pi r \sqrt{r^2 - x^2}}$$

$$f(x) = \boxed{\frac{1}{\pi \sqrt{r^2 - x^2}}}$$



Properly normalized?

$$1 \stackrel{?}{=} \int_{-r}^r f(x) dx = \int_{-r}^r \frac{1}{\pi \sqrt{r^2 - x^2}} dx = \frac{1}{\pi} \arcsin \left( \frac{x}{r} \right) \Big|_{-r}^r$$

$$= \frac{1}{\pi} \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) = \frac{1}{\pi} (\pi)$$

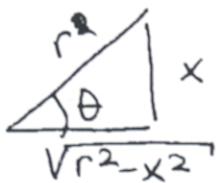
!      Indeed it is!

b.)  $\langle x \rangle = \int_{-r}^r x f(x) dx = \int_{-r}^r x \frac{1}{\pi \sqrt{r^2 - x^2}} dx$

$\uparrow$  odd       $\uparrow$  even  
 $= \boxed{0}$

$$\langle x^2 \rangle = \int_{-r}^r x^2 f(x) dx = \int_{-r}^r x^2 \frac{1}{\pi \sqrt{r^2 - x^2}} dx$$

( I see no other way, then a trig. subst )



$$x = r \sin \theta \Rightarrow dx = r \cos \theta d\theta$$

$$\sqrt{r^2 - x^2} = r \cos \theta$$

$$\begin{aligned}\Rightarrow \langle x^2 \rangle &= 2 \int_0^r \frac{x^2 dx}{\pi \sqrt{r^2 - x^2}} = 2 \int_0^{\pi/2} \frac{r^2 \sin^2 \theta \cdot r \cos \theta d\theta}{\pi r \cos \theta} \\ &= \frac{2r^2}{\pi} \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{2r^2}{\pi} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\ &= \boxed{\frac{r^2}{2}}\end{aligned}$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{r^4}{4}$$

$$\boxed{\sigma = \frac{r^2}{2}}$$

We could have noted that  $x = r \cos \theta$

$$\Rightarrow \langle x \rangle = \langle r \cos \theta \rangle = r \langle \cos \theta \rangle$$

$$\text{and } \langle x^2 \rangle = \langle r^2 \cos^2 \theta \rangle = r^2 \langle \cos^2 \theta \rangle$$

1.0

4.

$$f(x) = A e^{-\lambda(x-a)^2}$$

a.  $\int_{-\infty}^{\infty} dx f(x) = 1$

$$\int_{-\infty}^{\infty} dx A e^{-\lambda(x-a)^2} \stackrel{x' = x-a}{=} A \int_{-\infty}^{\infty} dx' e^{-\lambda x'^2}$$

$$= A \sqrt{\frac{\pi}{\lambda}} = 1$$

$$\Rightarrow A = \boxed{\sqrt{\frac{\lambda}{\pi}}}$$

b.  $\langle x \rangle = \int_{-\infty}^{\infty} \sqrt{\frac{\lambda}{\pi}} x e^{-\lambda(x-a)^2} dx$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{\lambda}{\pi}} (x+a) e^{-\lambda x'^2} dx'$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{\lambda}{\pi}} a e^{-\lambda x'^2} dx' + \int_{-\infty}^{\infty} x' \sqrt{\frac{\lambda}{\pi}} e^{-\lambda x'^2} dx'$$

$\uparrow$  even       $\uparrow$  odd

$$= a \sqrt{\frac{\lambda}{\pi}} \cdot \sqrt{\frac{\pi}{\lambda}} = a$$

$$\boxed{\langle x \rangle = a}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \sqrt{\frac{\lambda}{\pi}} x^2 e^{-\lambda(x-a)^2} dx$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{\lambda}{\pi}} (x'+a)^2 e^{-\lambda x'^2} dx'$$

$$x=x' \int_{-\infty}^{\lambda} \sqrt{\frac{\lambda}{\pi}} (x^2 + 2\overrightarrow{ax} + a^2) e^{-\lambda x^2} dx$$

$$= \sqrt{\frac{\lambda}{\pi}} \left[ \int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx + \int_{-\infty}^{\infty} a^2 e^{-\lambda x^2} dx \right]$$

$$\text{trick} = \sqrt{\frac{\lambda}{\pi}} \left[ -\frac{\partial}{\partial \lambda} \left( \int_{-\infty}^{\infty} e^{-\lambda x^2} dx \right) + a^2 \sqrt{\frac{\pi}{\lambda}} \right]$$

$$= \sqrt{\frac{\lambda}{\pi}} \left[ -\frac{\partial}{\partial \lambda} \left( \sqrt{\frac{\pi}{\lambda}} \right) + a^2 \sqrt{\frac{\pi}{\lambda}} \right]$$

$$= a^2 + \sqrt{\frac{\lambda}{\pi}} \cdot \sqrt{\pi} \cdot \left( + \frac{1}{2} \lambda^{-\frac{3}{2}} \right) = a^2 + \frac{1}{2\lambda}$$

$$\langle x^2 \rangle = a^2 + \frac{1}{2\lambda}$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = a^2 + \frac{1}{2\lambda} - a^2$$

$$\sigma^2 = \frac{1}{2\lambda}$$

$$\boxed{\sigma = \frac{1}{\sqrt{2\lambda}}}$$

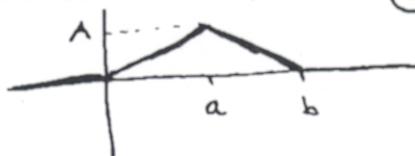
c.) looks like a bell:



1.7

6.

$$\psi(x, 0) =$$



(b)

a.)  $I = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_0^a (Ax/a)^2 dx + \int_a^b \left(\frac{A(b-x)}{(b-a)}\right)^2 dx$

 $= \frac{A^2}{a^2} \int_0^a x^2 dx + \frac{A^2}{(b-a)^2} \int_a^b (b-x)^2 dx$ 
 $= \frac{A^2}{a^2} \cdot \frac{1}{3} a^3 + \frac{A^2}{(b-a)^2} \left(-\frac{1}{3}(b-x)^3\right) \Big|_a^b$ 
 $= \frac{A^2}{3} a + \frac{A^2}{(b-a)^2} \cdot \frac{1}{3} (b-a)^3$ 
 $= \frac{A^2}{3} (a + (b-a)) = \frac{A^2}{3} b$

$$\frac{A^2}{3} b = 1 \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}}$$

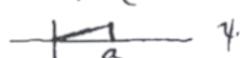
c.)  $|\psi(x)|^2$  has a maximum at  $x=a$ .  
 $\Rightarrow$  most likely at  $x=a$ .

d.)  $P(0 < x < a) = \int_0^a |\psi(x)|^2 dx$

$= \int_0^a \frac{3}{b} \frac{x^2}{a^2} dx = \frac{3}{ba^2} \cdot \frac{1}{3} x^3 \Big|_0^a = \frac{a}{b}$

$$\boxed{P(0 < x < a) = \frac{a}{b}}$$

$a=b \Rightarrow P=1$  (makes sense)



$b=2a \Rightarrow P=1/2$  (also makes sense)



$$\text{eg} \quad \langle x \rangle = \int_{-\infty}^{\infty} x \|\psi(x)\|^2 dx$$

$$\begin{aligned}
&= \int_0^a \frac{\frac{3}{b}}{b} \frac{x^2}{a^2} x dx + \int_b^a \frac{3}{b(b-a)^2} (b-x)^2 x dx \\
&= \frac{3}{ba^2} \int_0^a x^3 dx + \frac{3}{b(b-a)^2} \int_a^b (b^2x - 2bx^2 + x^3) dx \\
&= \frac{3}{ba^2} \left[ \frac{1}{4}a^4 \right] + \frac{3}{b(b-a)^2} \left( \frac{1}{2}b^2x^2 - \frac{2}{3}bx^3 + \frac{1}{4}x^4 \right) \Big|_a^b \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left( \frac{b^4}{12} - \frac{b^2a^2}{2} + \frac{2ba^3}{3} - \frac{a^4}{4} \right) \\
&= \frac{3a^2(b-a)^2 + b^4 - 6b^2a^2 + 8ba^3 - 3a^4}{4b(b-a)^2} \\
&= \frac{3a^2b^2 - 6a^3b + 3a^4 + b^4 - 6b^2a^2 + 8ba^3 - 3a^4}{4b(b-a)^2} \\
&= \frac{b^4 - 3a^2b^2 + 2a^3b}{4b(b-a)^2} = \frac{b^3 - 3a^2b + 2a^3}{4(b-a)^2} \\
&= \frac{(b-a)^2(b+2a)}{4(b-a)^2} = \frac{1}{4}(b+2a)
\end{aligned}$$

$$\boxed{\langle x \rangle = \frac{1}{4}(b+2a)}$$

1.5

8.

$$\begin{aligned}
 a.) \quad I &= \int_{-\infty}^{\infty} \|\psi(x,t)\|^2 dx \\
 &= \int_{-\infty}^{\infty} \psi^*(x,t) \psi(x,t) dx = \int_{-\infty}^{\infty} (A e^{-\lambda|x|} e^{-i\omega t})^* (A e^{-\lambda|x|} e^{-i\omega t}) dx \\
 &= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx \stackrel{\text{even}}{=} 2A^2 \int_0^{\infty} e^{-2\lambda x} dx \\
 &= 2A^2 \left( \frac{-1}{2\lambda} e^{-2\lambda x} \right) \Big|_0^{\infty} = 2A^2 \cdot \frac{1}{2\lambda} = \frac{A^2}{\lambda}
 \end{aligned}$$

$$\Rightarrow A^2 = \lambda$$

$$A = \sqrt{\lambda}$$

$$\begin{aligned}
 b.) \quad \langle x \rangle &= \int \|\psi(x,t)\|^2 x dx = A^2 \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx \\
 &= \boxed{0}
 \end{aligned}$$

$$\begin{aligned}
 \langle x^2 \rangle &= \int \|\psi(x,t)\|^2 x^2 dx = \lambda \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx \\
 &= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x} dx = -x^2 e^{-2\lambda x} \Big|_0^{\infty} + \int_0^{\infty} 2x e^{-2\lambda x} dx \\
 &= \int_0^{\infty} 2x e^{-2\lambda x} dx = \frac{-1}{\lambda} e^{-2\lambda x} x \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{\lambda} e^{-2\lambda x} dx \\
 &= \frac{1}{\lambda} \int_0^{\infty} e^{-2\lambda x} dx = \frac{1}{\lambda} \cdot \frac{-1}{2\lambda} e^{-2\lambda x} \Big|_0^{\infty} = \frac{1}{2\lambda^2}
 \end{aligned}$$

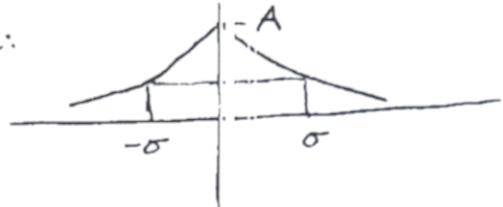
$$\boxed{\langle x^2 \rangle = \frac{1}{2\lambda^2}}$$

$$\begin{aligned} \sigma^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\ &= \frac{1}{2\lambda^2} \end{aligned}$$

$$\sigma = \frac{1}{\lambda} \cdot \frac{1}{2}\sqrt{2}$$

$$\boxed{\sigma = \frac{1}{2\lambda} \sqrt{2}}$$

Sketch:



$$P((x < -\sigma) \text{ OR } (x > \sigma))$$

$$\stackrel{\text{symmetry}}{=} 2P(x > \sigma) = 2 \int_{\frac{1}{2\lambda}\sqrt{2}}^{\infty} |\psi(x, t)|^2 dx.$$

$$= 2 \int_{\frac{1}{2\lambda}\sqrt{2}}^{\infty} A^2 e^{-2\lambda x} dx = 2\lambda \left[ \frac{e^{-2\lambda x}}{-2\lambda} \right]_{\frac{1}{2\lambda}\sqrt{2}}^{\infty}$$

$$= e^{-2\lambda(\frac{1}{2\lambda}\sqrt{2})} = \boxed{e^{-\sqrt{2}}}$$

$$\approx 24\%$$

log  
(a)

$$P_{ab} = \int_a^b |\psi(x, t)|^2 dx = \int_a^b \psi^*(x, t) \psi(x, t) dx$$

$$\frac{dP_{ab}}{dt} = \frac{d}{dt} \int \psi^*(x, t) \psi(x, t) dx = \int \left( \frac{d}{dt} \psi^* \right) \psi + \psi^* \left( \frac{d}{dt} \psi \right) dx$$

$$= \int \left[ -\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V \psi^* \right] \psi + \psi^* \left[ \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V \psi \right] dx$$

$$= \int_a^b \frac{i\hbar}{2m} \left( -\frac{\partial^2 \psi^*}{\partial x^2} \psi + \psi^* \frac{\partial^2 \psi}{\partial x^2} \right) dx$$

$$\stackrel{!}{=} \frac{i\hbar}{2m} \int_a^b \frac{\partial}{\partial x} \left( -\frac{\partial \psi^*}{\partial x} \psi + \psi^* \frac{\partial \psi}{\partial x} \right) dx$$

$$= \frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right) \Big|_a^b$$

use  $J(x,t)$  from the book

$$= \cancel{\dots} - J(x,t) \Big|_a^b$$

$$= -J(b,t) + J(a,t)$$

$$\stackrel{!}{=} J(a,t) - J(b,t) \quad \text{q.e.d.}$$

Units of  $J(x,t)$  is  $\text{s}^{-1}$

$$\text{b. } J(x,t) = \frac{i\hbar}{2m} \left( \psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right)$$

$$= \frac{i\hbar A}{2m} \left( e^{-\lambda|x|} e^{-iwt} \frac{\partial}{\partial x} (e^{-\lambda|x|} e^{iwt}) - e^{-\lambda|x|} e^{iwt} \frac{\partial}{\partial x} (e^{-\lambda|x|} e^{-iwt}) \right)$$

$$= \frac{i\hbar A}{2m} \left( e^{-\lambda|x|} \frac{\partial}{\partial x} (e^{-\lambda|x|}) - e^{-\lambda|x|} \frac{\partial}{\partial x} (e^{-\lambda|x|}) \right)$$

$$= \boxed{0}$$

III.

a)  $\frac{d}{dt} \int_{-\infty}^{\infty} \psi^*(x,t) \psi(x,t) dx = \int_{-\infty}^{\infty} \left[ \left( \frac{\partial}{\partial t} \psi^* \right) \psi + \psi^* \left( \frac{\partial}{\partial t} \psi \right) \right] dx$

$$= \int_{-\infty}^{\infty} \left( -\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} (V_0 + i\Gamma) \psi^* \right) \psi dx$$

$$+ \int_{-\infty}^{\infty} \psi^* \left( \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} (V_0 - i\Gamma) \psi \right) dx$$

$$= \int_{-\infty}^{\infty} \left[ \frac{i\hbar}{2m} \left( \psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2} \psi \right) - \frac{2\Gamma}{\hbar} \psi^* \psi \right] dx$$

$$= \int_{-\infty}^{\infty} \frac{i\hbar}{2m} \frac{\partial}{\partial x} \left( \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right) dx - \frac{2\Gamma}{\hbar} \int \psi^*(x,t) \psi(x,t) dx$$

$$= \left. \frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right) \right|_{-\infty}^{\infty} - \frac{2\Gamma}{\hbar} P(t)$$

Because the  $\psi$  vanishes at  $\infty$  the first term is zero.

$$\Rightarrow \boxed{\frac{d}{dt} P(t) = -\frac{2\Gamma}{\hbar} P(t)}$$

b)  $\frac{d}{dt} P(t) = -\frac{2\Gamma}{\hbar} P(t)$

$$\frac{1}{P(t)} \frac{d}{dt} P(t) = -\frac{2\Gamma}{\hbar}$$

$$\frac{d}{dt} \log P(t) = -\frac{2\Gamma}{\hbar}$$

$$\log P(t) = -\frac{2\Gamma}{\hbar} t + \tilde{c}$$

$$\boxed{P(t) = C e^{-\frac{2\Gamma}{\hbar} t}} \Rightarrow \boxed{\tau = \frac{\hbar}{2\Gamma}}$$